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A NUMERICAL METHOD FOR EVALUATING STRESSES IN GUN TUBES

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INTRODUCTION

The stress distribution in gun tubes is generally computed by using the classical Lamé solution for thick walled cylinders. Thermal stresses are calculated for temperature distributions that are assumed to be logarithmic [1]. This approach does not allow for consideration of the effects of changes in cross section nor for the condition that the internal pressure is applied over a limited and changing region of the tube. Neither does it permit the consideration of transient temperature distribution that may occur in a firing cycle [2]. The effects of non-uniform conditions along the axis and a transient temperature distribution changes the problem from the classical one dimensional problem of Lamé to a problem in two dimensions that does not in general have an analytical solution.

An approach with some success for finding numerical solutions to two dimensional problems in elasticity is the use of finite differences. Southwell [3] was the first to do this successfully by using a relaxation technique. More recently methods using block relaxation and direct solution of the system of difference equations have been used [4, 5, 6, 7, 8].

In this paper, a direct solution for the system of difference equations is used. Special handling of the coefficient matrix in the computation routine makes possible the handling of very large systems of equations.

[1] Numbers refer to Bibliography

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DEVELOPMENT OF EQUATIONS:

The differential equations governing the behavior of a gun tube are developed in terms of the radial and axial displacement which are designated u and w respectively.

The equations of equilibrium for circular symmetry in terms of stress and with no body forces are: [10]

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (1)$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0 \qquad (2)$$

Figure 1 shows a cross-section of a typical tube with pressure loading. The coordinate axis and stress notations are given.

The stress strain relations are assumed to be linear and are given by Hookes law.

$$\begin{aligned} \sigma_r &= \lambda e + 2G\varepsilon_r - (3\lambda + 2G)A \\ \sigma_\theta &= \lambda e + 2G\varepsilon_\theta - (3\lambda + 2G)A \\ \sigma_z &= \lambda e + 2G\varepsilon_z - (3\lambda + 2G)A \\ \tau_{rz} &= G\gamma_{rz} \end{aligned} \qquad (3)$$

where

$$A = \int_{T_0}^{T} \alpha(\theta)\, d\Theta$$

λ and G are Lame constants and α is the coefficient of linear expansion. Each is assumed to be temperature dependent. T_0 and T are the initial and current temperatures respectively.

The strains are defined in terms of displacements by,

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}, \quad \varepsilon_z = \frac{\partial w}{\partial z} + \varepsilon_z$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad e = \varepsilon_r + \varepsilon_\theta \qquad (4)$$

Dimensionless variables are defined by the expressions

$$\begin{aligned} u &= a\bar{u} \\ w &= a\bar{w} \\ r &= a\xi \qquad \xi_0 \le \xi \le 1 \\ z &= b\eta \qquad 0 \le \eta \le 1 \end{aligned} \qquad (5)$$

where ξ_0 depends on the bore radius, a is a characteristic radius and b the length of the tube. Equations (3), (4) and (5) are substituted into equations (1) and (2). The following two equations in $\bar{u}$ and $\bar{w}$ are obtained.

$$\frac{a}{b}\frac{\lambda + G}{\lambda + 2G}\frac{\partial^2 \bar{w}}{\partial\xi\partial\eta} + \frac{a}{b(\lambda + 2G)}\frac{\partial G}{\partial\xi}\frac{\partial \bar{w}}{\partial\xi} + \frac{a}{b(\lambda + 2G)}\frac{\partial\lambda}{\partial\xi}$$

$$\frac{\partial\bar{w}}{\partial\eta} + \frac{\partial^2\bar{u}}{\partial\xi^2} + \frac{a^2}{b^2}\frac{G}{(\lambda + 2G)}\frac{\partial^2\bar{u}}{\partial\eta^2} + \frac{1}{\lambda + 2G} \times$$

$$\left(\frac{\lambda}{\xi} + \frac{\partial\lambda}{\partial\xi} + \frac{2G}{\xi} + 2\frac{\partial G}{\partial\xi}\right)\frac{\partial\bar{u}}{\partial\xi} + \frac{a^2}{b^2(\lambda + 2G)}\frac{\partial G}{\partial\eta}\cdot\frac{\partial\bar{u}}{\partial\eta} +$$

$$\frac{1}{\lambda + 2G}\left(\frac{1}{\xi}\frac{\partial\lambda}{\partial\xi} - \frac{\lambda}{\xi^2} - \frac{2G}{\xi^2}\right)\bar{u} = \frac{1}{\lambda + 2G}\frac{\partial}{\partial\xi}[(3\lambda + 2G)A] \tag{6}$$

$$\frac{G}{\lambda + 2G}\frac{\partial^2\bar{w}}{\partial\xi^2} + \frac{a^2}{b^2}\frac{\partial^2\bar{w}}{\partial\eta^2} + \frac{1}{\lambda + 2G}\left(\frac{G}{\xi} + \frac{\partial G}{\partial\xi}\right)\frac{\partial\bar{w}}{\partial\xi} +$$

$$\frac{a^2}{b^2(\lambda + 2G)}\left(\frac{\partial\lambda}{\partial\eta} + 2\frac{\partial G}{\partial\eta}\right)\frac{\partial\bar{w}}{\partial\eta} + \frac{a(\lambda + G)}{b(\lambda + 2G)}\frac{\partial^2\bar{u}}{\partial\xi\partial\eta} +$$

$$\frac{a}{b(\lambda + 2G)}\frac{\partial\lambda}{\partial\eta}\frac{\partial\bar{u}}{\partial\xi} + \frac{a}{b(\lambda + 2G)}\left(\frac{\lambda}{\xi} + \frac{G}{\xi} + \frac{\partial G}{\partial\xi}\right)\frac{\partial\bar{u}}{\partial\eta}$$

$$+ \frac{a}{b(\lambda + 2G)}\frac{1}{\xi}\frac{\partial\lambda}{\partial\eta}\bar{u} = \frac{a}{b(\lambda + 2G)}\frac{\partial}{\partial\eta}[(3\lambda + 2G)A] \tag{7}$$

Equations (6) and (7), must be satisfied throughout the tube and are solved subject to the appropriate conditions on the boundary.

FINITE DIFFERENCE ANALOG

The finite difference analog for the differential equations (6) and (7) is obtained by replacing derivatives with finite difference approximations. To minimize the truncation error, a central difference approximation is used throughout. The dimensionless spacing on ξ is $h = 1/M$, and on η it is $k = 1/N$. The first, second and cross derivatives at a nodal point (i, j) in the open region of the tube cross section are:

$$\frac{\partial \bar{u}}{\partial \xi} = \frac{1}{2k} \left(\bar{u}_{i,j+1} - \bar{u}_{i,j-1}\right) \tag{8}$$

$$\frac{\partial^2 \bar{u}}{\partial \xi^2} = \frac{1}{k^2} \left(\bar{u}_{i,j+1} - 2\bar{u}_{i,j} + \bar{u}_{i,j-1}\right) \tag{9}$$

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{1}{4hk} \left(\bar{u}_{i+1,j+1} - \bar{u}_{i-1,j+1} - \bar{u}_{i+1,j-1} + \bar{u}_{i-1,j-1}\right) \tag{10}$$

If approximations of the form in equations (8), (9) and (10) are substituted into equations (6) and (7), two algebraic equations of the following form are obtained for each nodal point within the region.

$$\begin{aligned} & C_{1,k}\, \bar{w}_{i-1,j-1} + C_{2,k}\, \bar{u}_{i-1,j-1} + C_{3,k}\, \bar{w}_{i-1,j} + C_{4,k}\, \bar{u}_{i-1,j} \\ + \; & C_{5,k}\, \bar{w}_{i-1,j+1} + C_{6,k}\, \bar{u}_{i-1,j+1} + C_{7,k}\, \bar{w}_{i,j-1} + C_{8,k}\, \bar{u}_{i,j-1} \\ + \; & C_{9,k}\, \bar{w}_{i,j} + C_{10,k}\, \bar{u}_{i,j} + C_{11,k}\, \bar{w}_{i,j+1} + C_{12,k}\, \bar{u}_{i,j+1} \\ + \; & C_{13,k}\, \bar{w}_{i+1,j-1} + C_{14,k}\, \bar{u}_{i+1,j-1} + C_{15,k}\, \bar{w}_{i+1,j} \\ + \; & C_{16,k}\, \bar{u}_{i+1,j} + C_{17,k}\, \bar{w}_{i+1,j+1} + C_{18,k}\, \bar{u}_{i+1,j+1} \\ = \; & C_{19,k} \qquad k = 1, 2 \end{aligned} \tag{11}$$

where the coefficients $C_{i,k}$ are functions of material properties, and will, because of their dependence on temperature, be functions of position. In the solution of the problem the material properties are approximated by polynomials in temperature and their derivatives with respect to the space variables are obtained by using a central difference approximation.

The set of equations (11) will be equal in number to twice the number of nodal points within the region. The equations will include values of $\bar{u}$ and $\bar{w}$ at each of the nodal points inside the region and also values of $\bar{u}$ and $\bar{w}$ at nodal points on the boundary line surrounding the region. At each nodal point on the boundary line, two algebraic equations which express the boundary conditions in finite difference form are written. When combined with equations (11) for the interior a solvable set of simultaneous equations in discrete values of $\bar{u}$ and $\bar{w}$ are obtained.

BOUNDARY EQUATIONS:

Conditions at each nodal point along the boundary will be in the

form of stress or displacement, and in some instances it may be possible to define a line of symmetry that can form a boundary line. Boundary equations are generated for each type of condition and the program is written so that at each boundary point any one of the three conditions can be prescribed.

To best accommodate the three possible conditions at the boundary the nodal spacing is chosen so that the actual boundary of the region always lies midway between nodal lines, as illustrated by figure 2. Two boundary equations must be written for each of the nodal points on the outer perimeter. The nature of the equations will be such as to describe the condition at the boundary that is closest to the nodal point. Examples chosen from the bottom side of the region (shown in figure 2) will be illustrated. Suppose first that displacements at the nodal line i along the bottom boundary are given and are denoted by u_i and w_i. Then from the central difference approximation,

$$\frac{a}{2}\left(\bar{w}_{i,1}+\bar{w}_{i,2}\right)=w_i \tag{12a}$$

$$\frac{a}{2}\left(\bar{u}_{i,1}+\bar{u}_{i,2}\right)=u_i \tag{12b}$$

If the stresses at the boundary corresponding to the i^{th} nodal line are given, then from equations (3) and (4)

$$\lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}\right)+2G\frac{\partial u}{\partial r}-(3\lambda+2G)\Lambda=\sigma_r \tag{13a}$$

$$G\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)=\tau_{rz} \tag{13b}$$

The following difference equations follow,

$$\frac{a}{4bh}\left(\bar{w}_{i+1,2}-\bar{w}_{i-1,2}-\bar{w}_{i-1,1}+\bar{w}_{i+1,1}\right)+\frac{1}{k}\left(1+\frac{2G}{\lambda}\right)$$
$$\times\left(\bar{u}_{i,2}-\bar{u}_{i,1}\right)+\frac{1}{2(\xi_0+\frac{\lambda}{2})}\left(u_{i,1}+u_{i,2}\right)=\frac{\sigma_{r,i}}{\lambda}+$$
$$\left(3+\frac{2G}{\lambda}\right)\Lambda \tag{14a}$$

$$\frac{a}{4bh}\left(\bar{u}_{i+1,2}+\bar{u}_{i+1,1}-\bar{u}_{i-1,2}-\bar{u}_{i-1,1}\right)+\frac{1}{k}\times$$
$$\left(\bar{w}_{i,2}-\bar{w}_{i,1}\right)=\frac{\tau_{rz}}{G} \tag{14b}$$

The symmetry condition along a boundary is given by a mirror reflection in that boundary. For the point at the i^{th} nodal line along the bottom boundary

$$\bar{w}_{i,2} - \bar{w}_{i,1} = 0 \tag{15a}$$

$$\bar{u}_{i,2} + \bar{u}_{i,1} = 0 \tag{15b}$$

Sets of equations similar to equations (12), (14) or (15) are written for each point along the nodal boundary line. An exception occurs for a corner point when displacement or stress is prescribed at the corner. For displacement given at a corner, the boundary equation is obtained by setting the displacement equal to the average of the four neighboring points. A stress condition at a corner is handled by prescribing the stress on the slant surface of a triangular section with sides h/2 by k/2 as illustrated in figure 3. Stresses σ_n and τ_{ns} are written in terms of the boundary stresses which are in turn approximated by finite differences. An interior corner such as shown in figure 4 is handled by requiring the normal stress on each boundary be satisfied. The shear stress in the corner is ignored. Each boundary equation is of a form that can be written in the general form of equation (11). Thus all equations of the region and the boundary can be written in the format of equation (11). Two equations are written at each nodal point, be it interior or boundary point, and this defines a system of algebraic equations that is equal in number to the number of discrete values of u and w.

SOLUTION OF ALGEBRAIC SYSTEM

The system of algebraic equations is arranged in an order prescribed by varying j and then i, i.e., the first two equations of the set are written for the nodal point at (1, 1), the next two for (1, 2) etc. Equations for points on nodal line i will in general involve points on nodal lines at i - 1 and i + 1. Thus, the equations arranged in the above order leads to a banded coefficient matrix on the unknowns $\bar{u}$ and $\bar{w}$. For executing the solution of this system on a digital computer the matrix is partitioned on the basis of values of i. Then, in an elimination sequence, it is possible to deal only with the three consecutive sub matrices that corresponds to equations for points on the nodal lines i - 1, i, and i + 1. These matrices are brought into core, the elimination operations performed and then the core updated for the next elimination sequence. This enables a computer with limited core capacity to handle very large systems without great sacrifice in time for loading and unloading core. It also has the effect of limiting only the nodal spacing in the radial direction and giving the capacity for very large spacing in the

axial or z directions. For the IBM 7044 system with 32k core memory and unlimited disk storage, a system with 30 nodal lines in the radial direction, two equations for each point, and any number in the axial direction can be handled.

The computational sequence is handled in three independent steps. First the coefficient matrix is generated. This involves computing the cofficients C_{ik} for each of the equations in the set and storing them in a way to take advantage of the relatively narrow band width of the matrix. Next, the equations are solved for the unknown values of $\bar{u}$ and $\bar{w}$ by Gaussian elimination. The elimination routine is carried out with only a portion of the coefficient matrix in core at any time.

Finally with the displacements known, a separate program evaluates the stresses by using central difference approximations for the strains.

NUMERICAL EXAMPLE:

The problem chosen for sample computation is a 30MM gun tube of very light weight design. Internal pressures are not high but the light weight barrel gets hot very quickly in a sustained burst and there is real danger of catastrophic failure. A sketch of a cross-section of the tube is given in figure 1.

The physical properties of the barrel are given below for ambient temperature. These have been taken as constants for the results reported here.

$$\lambda = 1.15 \times 10^7 \text{ psi}$$

$$G = 1.73 \times 10^7 \text{ psi}$$

$$\alpha = 6.0 \times 10^{-6} \text{ in/in/°F}$$

Computations have been made for the following cases.

1. Tube at ambient temperature with internal pressure of:

   a. 6000 psi (projectile at 21")

   b. 3700 psi (projectile at 36")

2. Tube at 8 milliseconds after the 20th round is fired with internal pressure of:

   a. 6000 psi

   b. 3700 psi

The transient temperature used for cases 2 (a) and (b) is given in figure 5.

Results of the stress computations are given in figures 6, 7, and 8. Figure 6 is a plot of the three normal stresses versus radius for cases (1a) and (2a) at a spot 21" from the breech. Figure 7 shows results for cases (1b) and (1c) at 29" from the breech. Conditions at different positions along the barrel are similar in general form but will differ in magnitude. This is illustrated in figure 8 where the tangential stress is plotted against distance from the breech for cases (1a) and (2a).

In general, the results are as expected. For the case of ambient temperature, the values are very nearly the same as given by the classical theory. For the transient temperature condition, numerical values have not been checked against other methods of calculation but the general trend of the tangential stress seems correct. Perhaps less expected is the very high axial stress in the tube shown by σ_z in the curves.

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BECKETT

1. Engineering Design Handbook, Gun Series, Gun Tubes, Army Materiel Command Pamphlet, AMCP 706-252, February, 1964.

2. Adams, D. E., Brown, W. R., et al, Design Studies of the XM140 Barrel, Cornell Aeronautical Lab Report, Buffalo, New York, February, 1967.

3. Southwell, R. V., Relaxation Methods in Theoretical Physics, Volume 2, Oxford University Press, London, 1956.

4. Griffin, D. S. and Varga, R. S., "Numerical Solution of Plane Elasticity Problems," J. Soc. Indust. Appl. Math., Volume II, No. 4, 1963.

5. White, G. N., Difference Equations for Plane Thermal Elasticity, LAMS-2745, October, 1962.

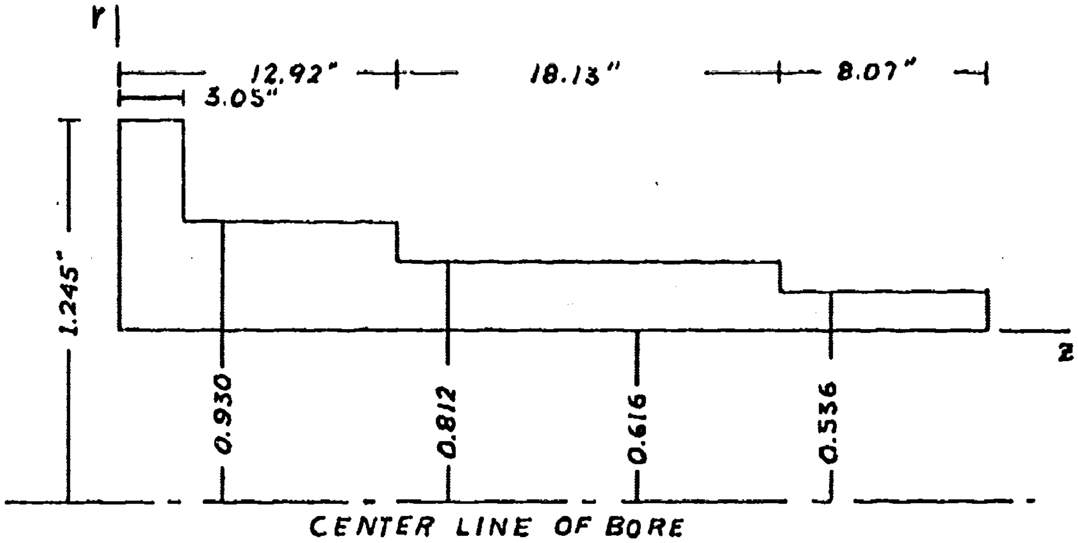
6. Griffin, D. S. and Kellog, R. B., A Numerical Solution for Axially Symmetrical and Plane Elasticity Problems, Unpublished Report, Westinghouse Electric Co., Pittsburgh, Pa.

7. Beckett, R. and Pan, K. C., Numerical Solution of Thermoelasticity Problems, Informal Report to Babcock & Wilcox Research Center, Alliance Ohio, 1967.

8. Forsythe, G. E. and Wasow, W. R., Finite-Difference Methods for Parnal Differential Equations, John Wiley and Sons, New York, 1960.

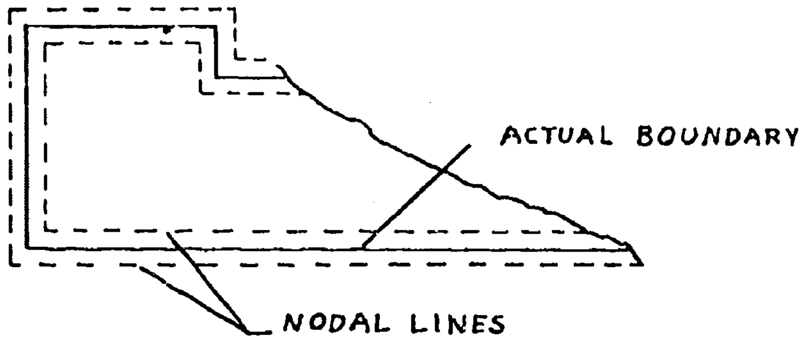
9. Beckett, R. and Hurt, J., Numerical Calculations and Algorithms, McGraw Hill, New York, 1966.

10. Timoshenko, S. and Goodier, J. N., Theory of Elasticity, McGraw Hill, New York, 1951.



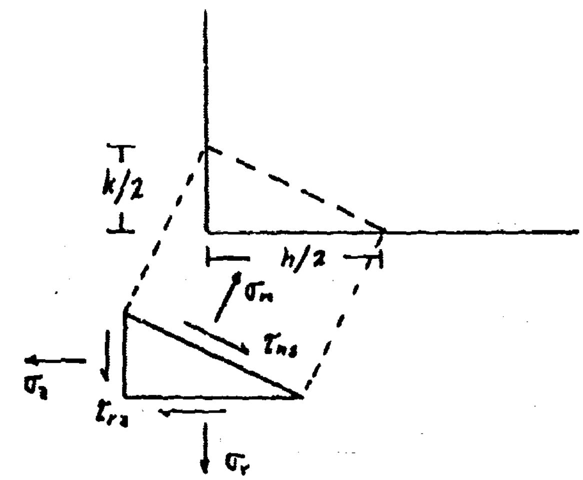
LONGITUDINAL SECTION OF TUBE

FIG. 1



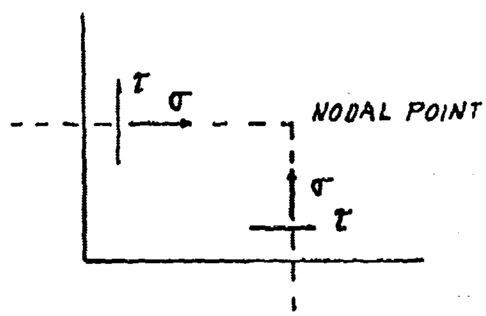
NODAL SPACING AT BOUNDARY

FIG. 2



EXTERIOR CORNER

FIG. 3



INTERIOR CORNER

FIG. 4

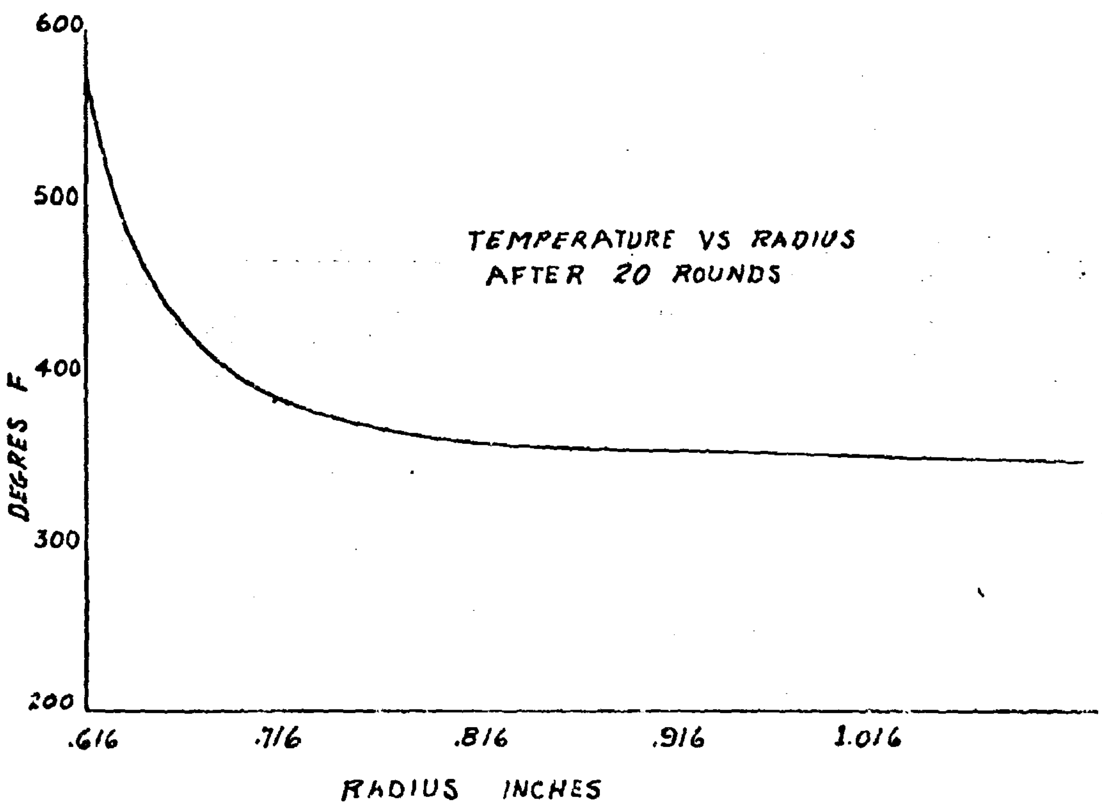
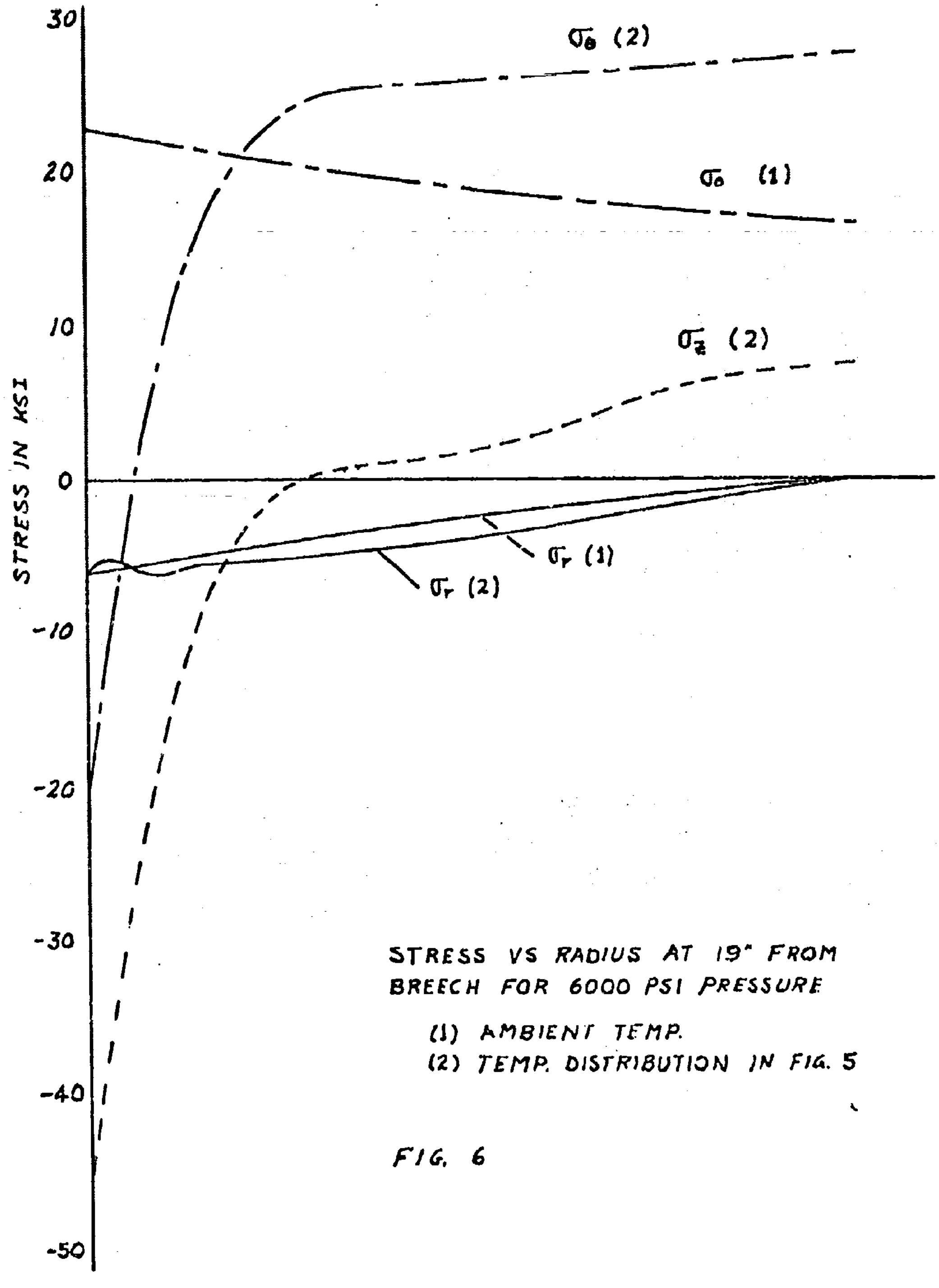


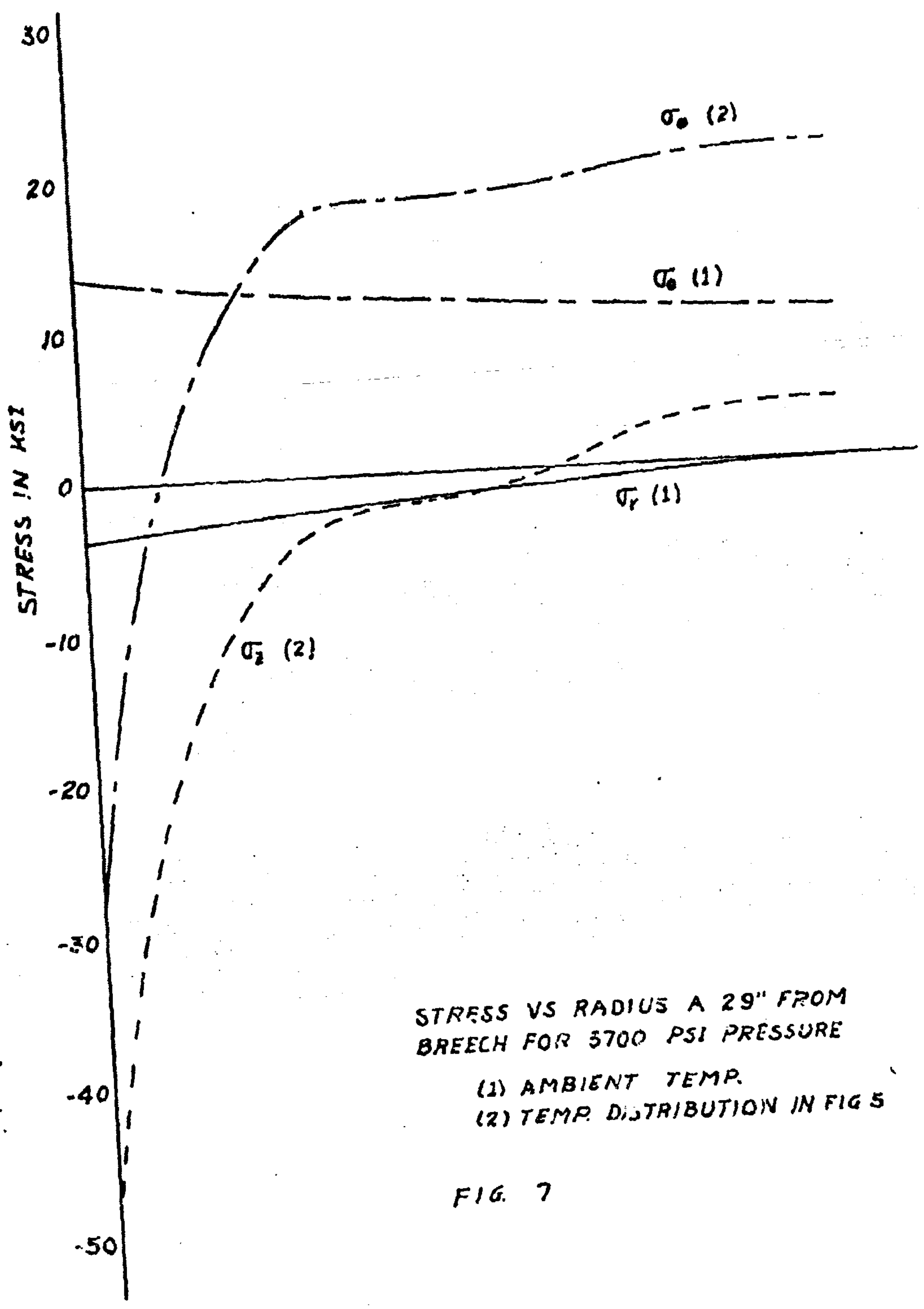
FIG. 5



STRESS VS RADIUS AT 19" FROM BREECH FOR 6000 PSI PRESSURE

(1) AMBIENT TEMP.
(2) TEMP. DISTRIBUTION IN FIG. 5

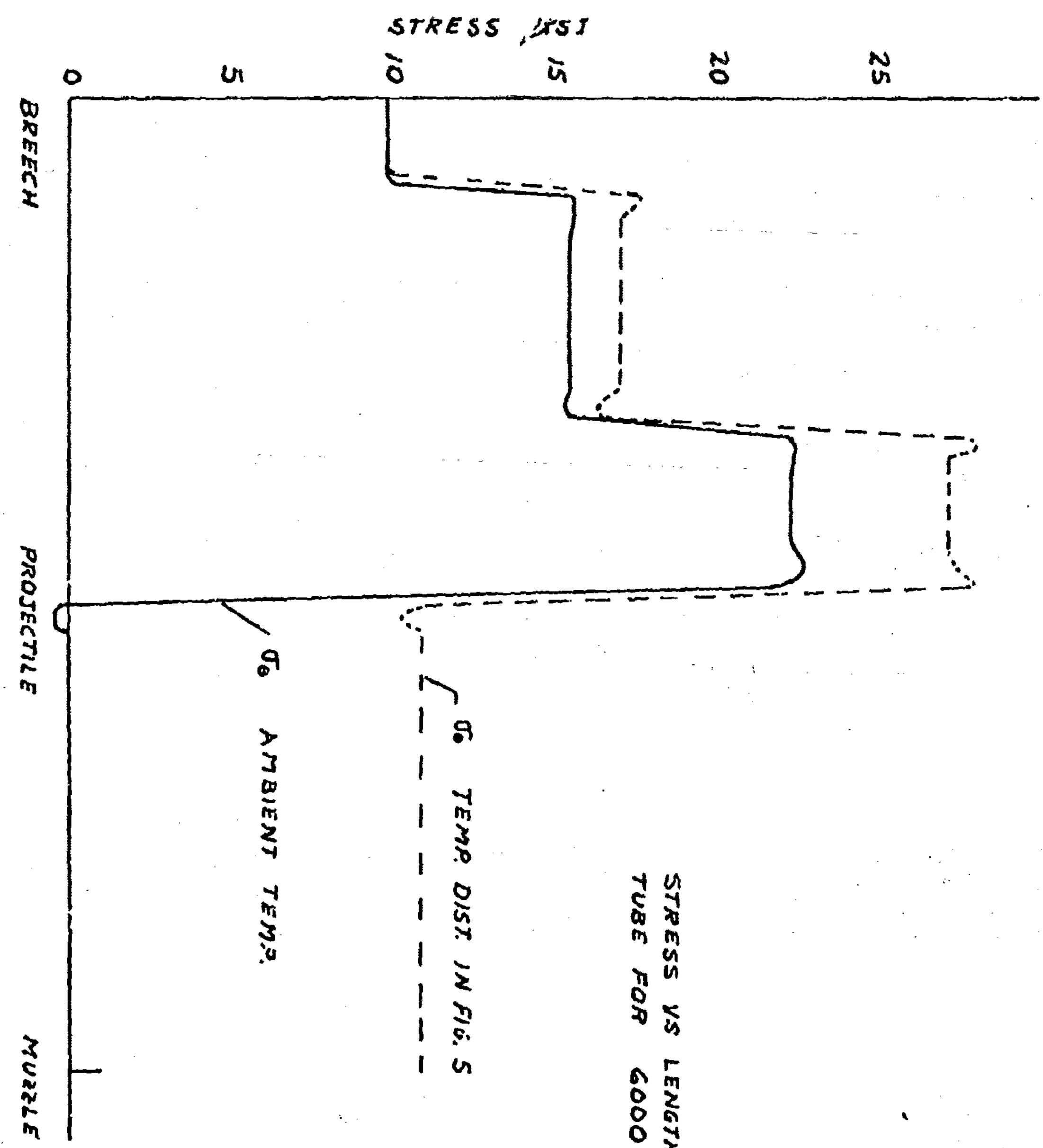
FIG. 6



STRESS VS RADIUS A 29" FROM BREECH FOR 5700 PSI PRESSURE

(1) AMBIENT TEMP.
(2) TEMP. DISTRIBUTION IN FIG 5

FIG. 7



STRESS VS LENGTH ALONG TUBE FOR 6000 PSI

FIG. 8